ON QUADRATIC COALGEBRAS, DUALITY AND THE UNIVERSAL STEENROD ALGEBRA

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Abstract. The notion of quadratic self-duality for coalgebras is developed with applications to algebraic structures which arise naturally in algebraic topology, related to the universal Steenrod algebra via an appropriate form of duality. This explains and unifies results of Lomonaco and Singer.

1. Introduction

In his work related to the cohomology of the Steenrod algebra, Singer [Sin83] introduced a graded coalgebra Γ over the field $\mathbb{F}_2$ and showed how to obtain the dual of the opposite of the Lambda algebra as a quotient of Γ. In related work [Sin05], Singer analysed the structure of a certain sub-coalgebra, which is dual to the Steenrod algebra of cohomology operations for Hopf algebra cohomology. These results can be compared with those of Lomonaco on the universal Steenrod algebra (aka. the big Steenrod algebra introduced by May in his general algebraic approach to the construction of Steenrod operations [May70]), who showed that the invariant theoretic constructions of Singer lead to a natural presentation of the universal Steenrod algebra as a quadratic algebra [Lom90]. There is a phenomenon of quadratic self-duality which relates these results; this is implicit in earlier results in the literature, where it appears under the guise of Koszul duality, for example in the results of Lomonaco on reciprocity [Lom92]; the relation with Singer's work [Sin05] has been investigated in [Lom06].

This note develops the general theory of quadratic self-duality and illustrates the theory by giving an independent construction of Singer's coalgebra Γ in the spirit of Milnor's description of the dual Steenrod algebra. As a biproduct of this approach, the Steenrod algebra appears by a process of group completion. A direct proof that the coalgebra is quadratic is given.

These results are then applied to the universal Steenrod algebra, which is identified as the associated quadratic algebra. An important fact is that, although the coalgebra Γ is not of finite type, it is obtained by a localization of a quadratic coalgebra $\mathscr{B}^+$, which is of finite type. An analogue of this result for the universal Steenrod algebra can be given.

The paper has two parts, the first considers generalities on quadratic self-duality and the second applies these results in the context of the universal Steenrod algebra. Section 2 reviews material on quadratic coalgebras, introducing the notion of admissibility (dual to the existence of monomial bases for quadratic algebras), a weak dual form of a PBW basis and the concept of transpose duality for admissible quadratic coalgebras. Quadratic self-duality is introduced in Section 3 both via vector space duality and using transpose duality. This is applied in Section 3.3 to prove the associated reciprocity results between pushforward and pullback quadratic coalgebras.

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The model for Γ is constructed in Section 4, using a monoid-valued functor on the category of commutative $\mathbb{F}_2$-algebras. The main result of the section is Theorem 4.5.4, which gives a direct proof that Γ is a quadratic coalgebra. The fundamental properties of Γ are established in Section 5, where it is shown that Γ is admissible and that Γ is quadratically self-dual (Theorem 5.3.1).

Finally, in Section 6, the connection with the universal Steenrod algebra is explained. The main result is Theorem 6.2.2, which shows that the universal Steenrod algebra is the quadratic algebra associated to Γ and, hence, is quadratically self-dual.

Contents

Part 1. **Quadratic self-duality**

2. Quadratic coalgebras

2.1. **One-cogenerated and quadratic coalgebras.** For the convenience of the reader, the basic notions concerning graded coalgebras over a field $\mathbb{K}$ (see [PV95], [MM65]).

The category of $\mathbb{N}$-graded vector spaces is symmetric monoidal with respect to the graded tensor product, with unit $\mathbb{K}$ concentrated in degree zero.

Definition 2.1.1. An $\mathbb{N}$-graded coalgebra is a counital comonoid in the category of $\mathbb{N}$-graded $\mathbb{K}$-vector spaces. A coalgebra C is connected if the counit ε induces an isomorphism $C_0 \cong \mathbb{K}$. A morphism of $\mathbb{N}$-graded coalgebras $C \to C'$ is a morphism of counital comonoids.

Notation 2.1.2. Let $\mathscr{E}_{\mathbb{K}}$ denote the category of $\mathbb{K}$-vector spaces and Coalg the category of $\mathbb{N}$-graded coalgebras and $\mathrm{Coalg}^{\mathrm{conn}} \subset \mathrm{Coalg}$ the full subcategory of connected coalgebras.

Let T_c denote the tensor coalgebra functor $T_c : \mathscr{E}_{\mathbb{K}} \to \mathrm{Coalg}^{\mathrm{conn}}$ which is right adjoint to the functor $\mathrm{Coalg}^{\mathrm{conn}} \to \mathscr{E}_{\mathbb{K}}$, which sends a graded coalgebra (C, Δ, ε) to C_1.

Definition 2.1.3.

(1) A graded connected coalgebra (C, Δ, ε) is one-cogenerated if the canonical morphism $C \to T_c(C_1)$ is a monomorphism of graded coalgebras.
(2) For V a $\mathbb{K}$-vector space and a subspace $R \leq V^{\otimes 2}$, let $\langle V; R\rangle$ denote the one-cogenerated graded coalgebra defined by

$$\langle V; R\rangle_n := \begin{cases} \mathbb{K} & n = 0 \\ V & n = 1 \\ \bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} & n \geq 2. \end{cases}$$

(3) A graded coalgebra C is quadratic if there exists a pair $V, R \leq V^{\otimes 2}$ and an isomorphism of graded coalgebras $C \cong \langle V; R\rangle$.
(4) Let $\mathrm{Coalg}^q \subset \mathrm{Coalg}^{\mathrm{conn}}$ denote the full subcategory of quadratic coalgebras.

Proposition 2.1.4. *The category* Coalg^q *is equivalent to the category of pairs of* $\mathbb{K}$*-vector spaces* $(V, R \leq V^{\otimes 2})$*, where a morphism* $(V, R \leq V^{\otimes 2}) \to (W, T \leq W^{\otimes 2})$ *is a morphism of vector spaces* $f : V \to W$ *such that* $f^{\otimes 2}$ *restricts to a morphism* $R \to T$.

Notation 2.1.5. For a pair of vector spaces $(V, R \leq V^{\otimes 2})$, let $\{V; R\}$ denote the associated quadratic algebra, defined as $T(V)/\langle R\rangle$, where $T(V)$ is the tensor algebra on V and $\langle R\rangle$ is the two-sided graded ideal generated by R.

Proposition 2.1.6. [PV95] *The category* Coalg^q *of quadratic coalgebras is equivalent to the category of quadratic algebras via the correspondence* $\langle V; R\rangle \leftrightarrow \{V; R\}$.

2.2. **Pullback and pushout constructions for quadratic coalgebras.** Fix a quadratic coalgebra $\langle V; R\rangle$.

Definition 2.2.1. For $W \leq V$ a sub-vector space, let $\langle W, R_W\rangle$ denote the quadratic coalgebra defined by $R_W := R \cap W^{\otimes 2}$.

Proposition 2.2.2. *Let* $i : W \hookrightarrow V$ *be an inclusion of vector spaces, then there is a unique monomorphism of quadratic coalgebras* $\langle W, R_W\rangle \hookrightarrow \langle V; R\rangle$ *which makes the following diagram commute:*

$$\begin{tikzcd}
\langle W; R_W\rangle \arrow[r, hook] \arrow[d, hook] & T_c(W) \arrow[d, hook, "T_c(i)"] \\
\langle V; R\rangle \arrow[r, hook] & T_c(V).
\end{tikzcd}$$

Remark 2.2.3. The underlying map $\langle W; R_W\rangle \hookrightarrow \langle V; R\rangle$ is a monomorphism of $\mathbb{N}$-graded $\mathbb{K}$-vector spaces.

Definition 2.2.4. Let $V \twoheadrightarrow Z$ be an epimorphism of vector spaces. The quadratic coalgebra $\langle Z; R^Z\rangle$ is the quadratic coalgebra defined by the image R^Z of R in $Z^{\otimes 2}$.

Proposition 2.2.5. *There is a unique morphism of coalgebras* $\langle V; R\rangle \to \langle Z; R^Z\rangle$ *which makes the following diagram commute*

$$\begin{tikzcd}
\langle V; R\rangle \arrow[r, hook] \arrow[d] & T_c(V) \arrow[d] \\
\langle Z; R^Z\rangle \arrow[r, hook] & T_c(Z).
\end{tikzcd}$$

Moreover, the morphism $\langle V; R\rangle \to \langle Z; R^Z\rangle$ *is an epimorphism of quadratic coalgebras.*

Remark 2.2.6. In general $\langle V; R\rangle \to \langle Z; R^Z\rangle$ is not a surjection of $\mathbb{N}$-graded $\mathbb{K}$-vector spaces.

2.3. **The quadratic dual coalgebra.** To give the usual definition of the quadratic dual coalgebra, a finiteness hypothesis is required.

Hypothesis 2.3.1. Let $\langle V; R\rangle$ be a quadratic coalgebra. Suppose that V is a $\mathbb{Z}$-graded $\mathbb{K}$-vector space, $R \leq V^{\otimes 2}$ is a graded (homogeneous) subspace and $V \otimes V$ is a vector space of finite type.

Notation 2.3.2.
(1) Let V^* denote the graded dual of the graded $\mathbb{K}$-vector space V.

(2) Let Coalg^q_{ft} denote the full subcategory of quadratic coalgebras satisfying Hypothesis 2.3.1 and let Alg^q_{ft} denote the analogous category of quadratic algebras of finite type.

Lemma 2.3.3. *Graded vector space duality induces an equivalence of categories*

$$\left(\mathrm{Coalg}^q_{ft}\right)^{\mathrm{op}} \xrightarrow{\cong} \mathrm{Alg}^q_{ft}.$$

Definition 2.3.4. Let $C := \langle V; R\rangle$ be a quadratic coalgebra satisfying hypothesis 2.3.1. The quadratic dual $C^!$ of C is the quadratic coalgebra $\{V^*, R^\perp\}$, where $R^\perp$ is the kernel of $V^* \otimes V^* \twoheadrightarrow R^*$.

Proposition 2.3.5. *Let $C := \langle V; R\rangle$ be an object of Coalg^q_{ft}.*

(1) *The quadratic coalgebra $C^!$ is the dual to the quadratic algebra $\{V; R\}$ (that is $\langle V; R\rangle^! \cong \{V; R\}^*$).*
(2) *There is a natural isomorphism of quadratic coalgebras $C \cong (C^!)^!$.*
(3) *Quadratic duality induces an equivalence of categories*

$$(\mathrm{Coalg}^q_{ft})^{\mathrm{op}} \xrightarrow{\cong} \mathrm{Coalg}^q_{ft}.$$

2.4. **Admissibility.**

Notation 2.4.1. Let $\mathbb{K}[\mathcal{I}]$ denote the free $\mathbb{K}$-vector space on the set $\mathcal{I}$. The canonical generator of $\mathbb{K}[\mathcal{I}]$ associated to the element $i \in \mathcal{I}$ is denoted by $[i]$.

To specify an isomorphism of $\mathbb{K}$-vector spaces $\mathbb{K}[\mathcal{I}] \xrightarrow{\cong} V$ is equivalent to giving a choice of basis indexed by $\mathcal{I}$.

Lemma 2.4.2. *Let $\mathcal{X}, \mathcal{Y}$ be sets.*

(1) *There are canonical isomorphisms $\mathbb{K}[\mathcal{X} \times \mathcal{Y}] \cong \mathbb{K}[\mathcal{X}] \otimes \mathbb{K}[\mathcal{Y}]$ and $\mathbb{K}[\mathcal{X} \amalg \mathcal{Y}] \cong \mathbb{K}[\mathcal{X}] \oplus \mathbb{K}[\mathcal{Y}]$ of vector spaces.*
(2) *There is a canonical monomorphism of $\mathbb{K}$-vector spaces $\mathbb{K}[\mathcal{X}] \hookrightarrow \mathbb{K}[\mathcal{X} \amalg \mathcal{Y}]$ and a canonical split epimorphism $\mathbb{K}[\mathcal{X} \amalg \mathcal{Y}] \twoheadrightarrow \mathbb{K}[\mathcal{X}]$ given by*

$$[w \in \mathcal{X} \amalg \mathcal{Y}] \mapsto \begin{cases} 0 & w \in \mathcal{Y} \\ [w \in \mathcal{S}] & w \in \mathcal{X}. \end{cases}$$

Notation 2.4.3. Write $\mathcal{S}'$ for the complement of a subset $\mathcal{S} \subset \mathcal{I} \times \mathcal{I}$.

Definition 2.4.4. Let $(\mathcal{I}, \mathcal{S} \subset \mathcal{I} \times \mathcal{I})$ be a pair of sets. The quadratic coalgebra $\langle V; R\rangle$ is $(\mathcal{I}, \mathcal{S})$-admissible if there exists an isomorphism $\mathbb{K}[\mathcal{I}] \xrightarrow{\cong} V$ such that the composite $R \to \mathbb{K}[\mathcal{S}]$ defined by the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\cong"'] & V \otimes V \arrow[d, "\cong"] \\ \mathbb{K}[\mathcal{S}] & \mathbb{K}[\mathcal{I} \times \mathcal{I}] \arrow[l, two heads] \end{tikzcd}$$

is an isomorphism, where the isomorphism $V \otimes V \cong \mathbb{K}[\mathcal{I} \times \mathcal{I}]$ is induced by the canonical isomorphism $\mathbb{K}[\mathcal{I} \times \mathcal{I}] \cong \mathbb{K}[\mathcal{I}] \otimes \mathbb{K}[\mathcal{I}]$ and the given isomorphism $\mathbb{K}[\mathcal{I}] \cong V$.

Remark 2.4.5. Compare [PP05, Chapter 4, Section 1] for conditions which establish the existence of admissible bases (in the case of quadratic algebras).

Proposition 2.4.6. *Let $(\mathcal{I}, \mathcal{S})$ be as above. The set of $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebras correspond bijectively to the set of functions*

$$f : \mathcal{S} \times \mathcal{S}' \to \mathbb{K}$$

such that, for each $s \in S$, the restriction $f(s, -) : \mathcal{S}' \to \mathbb{K}$ has finite support.

Proof. Suppose that $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ is an $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra. By hypothesis, there is an isomorphism $R \cong \mathbb{K}[\mathcal{S}]$ with respect to which the inclusion $R \hookrightarrow \mathbb{K}[\mathcal{I}] \otimes \mathbb{K}[\mathcal{I}] \cong \mathbb{K}[\mathcal{I}\times\mathcal{I}]$ is defined by

$$[s] \mapsto [s] - \Sigma_{s'\in\mathcal{S}'} f(s,s')[s'], \tag{1}$$

for some coefficients $f(s,s') \in \mathbb{K}$, such that the function $f(s,-) : s' \mapsto f(s,s')$ has finite support.

Conversely, given such a function $f : \mathcal{S}\times\mathcal{S}' \to \mathbb{K}$, the equation (1) defines an inclusion $\mathbb{K}[\mathcal{S}] \hookrightarrow \mathbb{K}[\mathcal{I}]\otimes\mathbb{K}[\mathcal{I}]$ which gives an $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra. □

Notation 2.4.7. Denote the $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ associated to the function $f : \mathcal{S}\times\mathcal{S}' \to \mathbb{K}$ by $\langle \mathcal{I}; \mathcal{S}, f\rangle$.

2.5. **Transpose duality.** For the purposes of this paper, a generalization of the quadratic duality functor is required, which allows the finite-type hypothesis to be relaxed.

Definition 2.5.1. The $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra $\langle \mathcal{I}; \mathcal{S}, f\rangle$ is dualizable if the function $f(-,s') : \mathcal{S} \to \mathbb{K}$ has finite support, for every $s' \in \mathcal{S}'$.

Notation 2.5.2. For $f : \mathcal{S}\times\mathcal{S}' \to \mathbb{K}$ a function, write $f^! : \mathcal{S}'\times\mathcal{S} \to \mathbb{K}$ for the function $f(s', s) = f(s,s')$, $\forall (s',s) \in \mathcal{S}'\times\mathcal{S}$.

Definition 2.5.3. The transpose dual of the dualizable $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra $\langle \mathcal{I}; \mathcal{S}, f\rangle$ is the quadratic coalgebra $\langle \mathcal{I}; \mathcal{S}, f\rangle^! := \langle \mathcal{I}; \mathcal{S}', f^!\rangle$.

Remark 2.5.4. The terminology transpose duality is used to avoid confusion with that of quadratic duality. The two notions can be related when $\mathcal{I}$ is equipped with a $\mathbb{Z}$-grading (that is a function $\mathcal{I} \to \mathbb{Z}$) and $\mathbb{K}[\mathcal{I}\times\mathcal{I}]$ is of finite type with respect to this grading. In this case, the dual basis provides a canonical isomorphism between $\mathbb{K}[\mathcal{I}\times\mathcal{I}]^*$ and $\mathbb{K}[\mathcal{I}\times\mathcal{I}]$ which relates the two notions.

2.6. **Admissibility, pullbacks and pushforwards.** In general, the property of admissibility is not preserved under pullbacks or pushforwards; the following result gives an explicit criterion when taking a sub-basis of cogenerators.

Notation 2.6.1. For $(\mathcal{I}, \mathcal{S} \subset \mathcal{I}\times\mathcal{I})$ sets and $\mathcal{J} \subset \mathcal{I}$ a subset, write $\mathcal{S}_{\mathcal{J}} := \mathcal{S} \cap \mathcal{J}^{\times 2}$.

Proposition 2.6.2. *Let $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ be an $(\mathcal{I},\mathcal{S})$-admissible quadratic coalgebra. For $\mathcal{J} \subset \mathcal{I}$ a subset, the pullback quadratic coalgebra $\langle \mathbb{K}[\mathcal{J}]; R_{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J},\mathcal{S}_{\mathcal{J}})$-admissible if and only if the function $f : \mathcal{S}\times\mathcal{S}' \to \mathbb{K}$ associated to $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ satisfies*

$$f(s_{\mathcal{J}}, s') = 0$$

for all $s_{\mathcal{J}} \in \mathcal{S}_{\mathcal{J}}$ and $s' \in \mathcal{S}'\backslash\mathcal{S}'_{\mathcal{J}}$.

Proof. Under the hypothesis on $\langle \mathbb{K}[\mathcal{I}]; R\rangle$, there is a commutative diagram:

$$\begin{tikzcd}
R_{\mathbb{K}[\mathcal{J}]} \arrow[r, hook] \arrow[d, hook] & \mathbb{K}[\mathcal{J}\times\mathcal{J}] \arrow[r, two heads] \arrow[d, hook] & \mathbb{K}[\mathcal{S}_{\mathcal{J}}] \arrow[d, hook] \\
R \arrow[r] \arrow[rr, bend right, "\cong"'] & \mathbb{K}[\mathcal{I}\times\mathcal{I}] \arrow[r] & \mathbb{K}[\mathcal{S}],
\end{tikzcd}$$

where the left hand square is a pullback (by definition) and the right hand square is defined using the canonical inclusions and projections of Lemma 2.4.2; commutativity is a consequence of the definition of $\mathcal{S}_{\mathcal{J}}$.

It follows that the composite of the top row is a monomorphism; commutativity of the left hand square shows that it is an isomorphism if and only if the given condition is satisfied. □

Similarly, for the pushforward of an admissible quadratic coalgebra:

Proposition 2.6.3. *Let $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ be an $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebra. For $\mathcal{J} \subset \mathcal{I}$ a subset, the pushforward quadratic coalgebra $\langle \mathbb{K}[\mathcal{J}]; R^{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J}, \mathcal{S}_\mathcal{J})$-admissible if and only if the function $f : \mathcal{S} \times \mathcal{S}' \to \mathbb{K}$ associated to $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ satisfies*

$$f(s, s') = 0$$

for all $s \in \mathcal{S}\backslash\mathcal{S}_\mathcal{J}$ and $s' \in \mathcal{S}'_\mathcal{J}$.

Proof. The proof is formally dual to that of Proposition 2.6.2. □

These conditions are related by transpose duality.

Corollary 2.6.4. *Let $\langle \mathbb{K}[\mathcal{I}]; R\rangle \cong \langle \mathcal{I}; \mathcal{S}, f\rangle$ be a dualizable quadratic coalgebra, with transpose dual $\langle \mathbb{K}[\mathcal{I}]; T\rangle \cong \langle \mathcal{I}; \mathcal{S}', f^!\rangle$ and $\mathcal{J} \subset \mathcal{I}$ be a subset. The following conditions are equivalent:*

1. *the pullback quadratic coalgebra $\langle \mathbb{K}[\mathcal{J}]; R_{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J}, \mathcal{S}_\mathcal{J})$-admissible;*
2. *the pushforward quadratic coalgebra $\langle \mathbb{K}[\mathcal{J}]; T^{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J}, \mathcal{S}'_\mathcal{J})$-admissible.*

Proof. The condition of Proposition 2.6.3 applied to the transpose dual (thus replacing $(f, \mathcal{S})$ by $(f^!, \mathcal{S}')$) is equivalent to the condition of Proposition 2.6.2. □

2.7. **The weak coPBW property and surjectivity.** In general, an epimorphism of quadratic graded coalgebras is not surjective as a morphism of graded vector spaces, (see Remark 2.2.6). In the presence of suitable bases, the surjectivity as a morphism of graded vector spaces can be established.

Definition 2.7.1. For $\mathcal{I}$ a set and $\mathcal{S} \subset \mathcal{I} \times \mathcal{I}$, let $\mathcal{S}^{(n)} \subset \mathcal{I}^n$ ($n \in \mathbb{N}$) be the subsets defined by $\mathcal{S}^{(0)} = \{\emptyset\}$, $\mathcal{S}^{(1)} = \mathcal{I}$ and, for $n \geq 2$, $\mathcal{S}^{(n)}$ recursively:

$$\mathcal{S}^{(n)} := (\mathcal{S}^{(n-1)} \times \mathcal{I}) \cap (\mathcal{I}^{\times n-2} \times \mathcal{S}).$$

The following definition is inspired by the notion of a PBW basis, introduced by Priddy [Pri70].

Definition 2.7.2. An $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ satisfies the weak coPBW property if, for all $n \in \mathbb{N}$, the following composite is an isomorphism

$$\begin{tikzcd} \langle \mathbb{K}[\mathcal{I}]; R\rangle_n \arrow[r, hook] \arrow[d, "\cong"'] & \mathbb{K}[\mathcal{I}]^{\otimes n} \arrow[d, "\cong"] \\ \mathbb{K}[\mathcal{S}^{(n)}] & \mathbb{K}[\mathcal{I}^{\times n}]. \arrow[l, two heads] \end{tikzcd}$$

(The condition is satisfied for $n \leq 2$ by the admissibility hypothesis.)

The following is clear:

Lemma 2.7.3. *For $(\mathcal{I}, \mathcal{S})$, $\mathcal{J}$ as above and $n \in \mathbb{N}$, $\mathcal{S}^{(n)}_\mathcal{J} \cong \mathcal{S}^{(n)} \cap \mathcal{J}^{\times n}$.*

Proposition 2.7.4. *Let $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ be an $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebra and $\mathcal{J} \subset \mathcal{I}$ be a subset for which $\langle \mathbb{K}[\mathcal{J}]; R^{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J}, \mathcal{S}_\mathcal{J})$-admissible.*

If the coalgebras $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ and $\langle \mathbb{K}[\mathcal{J}]; R^{\mathbb{K}[\mathcal{J}]}\rangle$ both satisfy the weak coPBW property, then the induced morphism of quadratic coalgebras

$$C := \langle \mathbb{K}[\mathcal{I}]; R\rangle \to C^\mathcal{J} := \langle \mathbb{K}[\mathcal{J}]; R^{\mathbb{K}[\mathcal{J}]}\rangle$$

is surjective as a morphism of $\mathbb{N}$-graded $\mathbb{K}$-vector spaces.

Proof. For $n \in \mathbb{N}$, the hypotheses ensure that there is a commutative diagram

$$\begin{tikzcd} C_n \arrow[r, hook] \arrow[rr, bend left, "\cong"] \arrow[d] & \mathbb{K}[\mathcal{I}^{\times n}] \arrow[r, two heads] \arrow[d] & \mathbb{K}[\mathcal{S}^{(n)}] \arrow[d] \\ C_n^{\mathcal{J}} \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & \mathbb{K}[\mathcal{J}^{\times n}] \arrow[r, two heads] & \mathbb{K}[\mathcal{S}^{(n)}_{\mathcal{J}}]. \end{tikzcd}$$

(The commutativity of the right hand square is a consequence of naturality of the projection and the commutativity of the left hand square follows from the construction of the morphism.) The result follows. □

3. Quadratic self-duality

The concept of quadratic self-duality is important; for instance, it leads to a bijection between the set of pullback quadratic coalgebras and the set of pushforward quadratic coalgebras of a quadratically self-dual coalgebra. Two flavours of self-duality are considered here: the standard approach via vector-space duality and an approach in the admissible setting which allows the finiteness hypotheses to be relaxed slightly.

3.1. **Self-duality via vector space duality.** To begin the discussion of quadratic self-duality, suppose that $\langle V; R\rangle$ is a quadratic coalgebra for which Hypothesis 2.3.1 holds.

Definition 3.1.1. The quadratic coalgebra $C := \langle V; R\rangle$ is quadratically self-dual if there exists an isomorphism $\varphi : V \overset{\cong}{\to} V^*$ which induces an isomorphism of quadratic coalgebras

$$C = \langle V; R\rangle \overset{\cong}{\to} C^! = \langle V^*; R^\perp\rangle.$$

This condition is equivalent to the assertion that φ induces a commutative diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\cong"'] & V^{\otimes 2} \arrow[d, "\varphi^{\otimes 2}"] \\ R^\perp \arrow[r, hook] & (V^*)^{\otimes 2}. \end{tikzcd}$$

Proposition 3.1.2.

(1) *Let $\langle V; R\rangle$ be a quadratically self-dual coalgebra, where $\dim V$ is finite. Then $2\dim R = (\dim V)^2$; in particular, $\dim V$ is even.*

(2) *A quadratic coalgebra $\langle V; R\rangle$ which satisfies Hypothesis 2.3.1 is quadratically self-dual if and only if $\langle V; R\rangle^!$ is quadratically self-dual.*

Proof. Straightforward. □

Example 3.1.3. Let $\mathbb{K}$ be a field and $\mathcal{I}$ be the set $\{x, y\}$.

(1) Consider the quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R\rangle$, where R is generated by $[x]\otimes[y]$ and $[y]\otimes[x]$. Writing η_x, η_y for the dual basis of $\mathbb{K}[\mathcal{I}]^* \cong \mathbb{K}^{\mathcal{I}}$, $R^\perp$ is generated by $\eta_x\otimes\eta_x, \eta_y\otimes\eta_y$. This quadratic coalgebra is not quadratically self-dual.

(2) The quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; T\rangle$, where T is generated by $[x]\otimes[x]$ and $[x]\otimes[y]$ has quadratic dual $\langle \mathbb{K}^{\mathcal{I}}; T^\perp\rangle$, where $T^\perp$ is generated by $\eta_y\otimes\eta_x$ and $\eta_y\otimes\eta_y$. The isomorphism φ defined by $x \mapsto \eta_y$ and $y \mapsto \eta_x$ shows that $\langle \mathbb{K}[\mathcal{I}]; T\rangle$ is quadratically self-dual.

(3) In [MOS09], the authors consider a $\mathbb{C}$-algebra A defined as the path algebra of a quiver; the algebra A is quadratically self-dual [MOS09, page 1132], hence dualizing gives a quadratically self-dual coalgebra.

Proposition 3.1.4. *Let $C := \langle V; R\rangle$ be a quadratic coalgebra which satisfies Hypothesis 2.3.1 and which is quadratically self-dual. Then the dual quadratic algebra C^* is isomorphic to the quadratic algebra $\{V; R\}$.*

Proof. Under the finiteness hypothesis, Proposition 2.3.5 implies that $C^!$ is dual to the quadratic algebra $\{V; R\}$. The quadratic self-duality yields an isomorphism of quadratic coalgebras $C \cong C^!$, which gives the result. □

3.2. **Strict self-duality.** In the case of dualizable admissible quadratic coalgebras, there is a strict version of quadratic self-duality, restricting the class of isomorphisms considered.

Definition 3.2.1. Let $\mathcal{I}$, $\mathcal{S} \subset \mathcal{I}^{\times 2}$ be sets and $C := \langle \mathcal{I}; \mathcal{S}, f\rangle$ be an $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebra which is dualizable.

The coalgebra C is strictly self-dual if there exists a bijection $\sigma : \mathcal{I} \stackrel{\cong}{\rightarrow} \mathcal{I}$ of sets such that

(1) σ restricts to a bijection $\sigma : \mathcal{S} \stackrel{\cong}{\rightarrow} \mathcal{S}'$;
(2) the function $f : \mathcal{S} \times \mathcal{S}' \rightarrow \mathbb{K}$ satisfies the identity $f = f^! \circ (\sigma \times \sigma)$.

Proposition 3.2.2. *Suppose that $\langle \mathcal{I}; \mathcal{S}, f\rangle$ is dualizable and strictly self-dual with respect to $\sigma : \mathcal{I} \stackrel{\cong}{\rightarrow} \mathcal{I}$, then $\mathbb{K}[\sigma] : \mathbb{K}[\mathcal{I}] \stackrel{\cong}{\rightarrow} \mathbb{K}[\mathcal{I}]$ induces an isomorphism of quadratic coalgebras $\langle \mathcal{I}; \mathcal{S}, f\rangle \stackrel{\cong}{\rightarrow} \langle \mathcal{I}; \mathcal{S}', f^!\rangle$.*

Proof. Clear. □

3.3. **Quadratic duality, pushforward and pullback.** Suppose that $\langle V; R\rangle$ satisfies Hypothesis 2.3.1. Consider a subspace $W \leq V$, which defines the restriction R_W and the commutative diagram

$$\begin{tikzcd}
R_W \arrow[r, hook] \arrow[d, hook] & W^{\otimes 2} \arrow[r, two heads] \arrow[d, hook] & W^{\otimes 2}/R_W \arrow[d, hook] \\
R \arrow[r, hook] & V^{\otimes 2} \arrow[r, two heads] & V^{\otimes 2}/R,
\end{tikzcd}$$

in which the right hand vertical morphism is a monomorphism, since the left hand square is a pullback.

Vector space duality yields the diagram

$$\begin{tikzcd}
R^{\perp} \arrow[r, hook] \arrow[d, two heads] & (V^*)^{\otimes 2} \arrow[r, two heads] \arrow[d, two heads] & R^* \arrow[d, two heads] \\
R_W^{\perp} \arrow[r, hook] & (W^*)^{\otimes 2} \arrow[r, two heads] & R_W^*
\end{tikzcd}$$

and hence a canonical morphism of quadratic coalgebras

$$\langle V; R\rangle^! = \langle V^*; R^{\perp}\rangle \rightarrow \langle W^*; R_W^{\perp}\rangle = \langle W; R_W\rangle^!.$$

Proposition 3.3.1. *Let $\langle V; R\rangle$ be a quadratic coalgebra which satisfies Hypothesis 2.3.1 and which is quadratically self-dual.*

(1) *For $W \leq V$, the quadratic duality $\varphi : V \stackrel{\cong}{\rightarrow} V^*$ induces a canonical surjection of quadratic coalgebras*

$$\langle V; R\rangle \rightarrow \langle W; R_W\rangle^!$$

with underlying morphism $V \stackrel{\varphi}{\rightarrow} V^* \rightarrow W^*$.

(2) *For $V \twoheadrightarrow Z$, the quadratic duality φ induces a canonical monomorphism of quadratic coalgebras*
$$\langle Z; R^Z \rangle^! \hookrightarrow \langle V; R \rangle$$
with underlying morphism $Z^* \to V^* \stackrel{\varphi^{-1}}{\to} V$.

(3) *These constructions are mutually inverse and define a bijection between the set of pullback quadratic coalgebras of $\langle V; R\rangle$ and the set of pushforward quadratic coalgebras of $\langle V; R\rangle$.*

Proof. The construction of the associated quotient is outlined above and the second construction is formally dual. □

In the case of admissible quadratic coalgebras, the above result has an analogue.

Proposition 3.3.2. *Let $\langle \mathbb{K}[\mathcal{I}]; R\rangle \cong \langle \mathcal{I}; \mathcal{S}, f\rangle$ be a dualizable $(\mathcal{I}, \mathcal{S})$-admissible quadratic coalgebra, which is strictly self-dual with respect to the bijection $\sigma : \mathcal{I} \stackrel{\cong}{\to} \mathcal{I}$. The following conditions on a set $\mathcal{J} \subset \mathcal{I}$ are equivalent:*

(1) *the pullback quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R_{\mathbb{K}[\mathcal{J}]}\rangle$ is $(\mathcal{J}, \mathcal{S}_{\mathcal{J}})$-admissible;*

(2) *the pushforward quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R^{\mathbb{K}[\sigma(\mathcal{J})]}\rangle$ is $(\mathcal{J}, \mathcal{S}_{\sigma(\mathcal{J})})$ admissible.*

Proof. The result follows directly from Corollary 2.6.4; it is nevertheless worthwhile to make the key step of the argument explicit.

By strict duality, $f(s, s') = f^!(\sigma(s), \sigma(s')) = f(\sigma(s'), \sigma(s))$, hence the following two conditions are equivalent:

- $f(s, s') = 0$ for all $s \in \mathcal{S}_{\mathcal{J}}$ and $s' \in \mathcal{S}' \backslash \mathcal{S}'_{\mathcal{J}}$;
- $f(s, s') = 0$ for all $s \in \mathcal{S} \backslash \mathcal{S}_{\sigma(\mathcal{J})}$ and $s' \in \mathcal{S}'_{\sigma(\mathcal{J})}$,

since σ induces a bijection between $\mathcal{S}$ and $\mathcal{S}'$.

The result follows by applying Propositions 2.6.2 and 2.6.3. □

Part 2. Applications related to the Steenrod algebra

4. Bialgebras associated to additive polynomials over $\mathbb{F}_2$

4.1. **Graded bialgebras.** The following notion corresponds to that of a coalgebra with products introduced by Singer in [Sin05, Definition 2.1], although a simpler terminology is preferred here, working over $\mathbb{F}_2$.

Definition 4.1.1. The category of $\mathbb{N}$-graded bialgebras over $\mathbb{F}_2$ is the category of $\mathbb{N}$-graded counital comonoids in the symmetric monoidal category of commutative graded $\mathbb{F}_2$-algebras.

Definition 4.1.2. An $\mathbb{N}$-graded bialgebra B is quadratic if the underlying $\mathbb{N}$-graded coalgebra is quadratic.

4.2. **Monoids associated to additive polynomials over $\mathbb{F}_2$.** The terminology graded monoid is used to denote an $\mathbb{N}$-graded monoid, so that the monoid is given by sets $\{M_n | n \in \mathbb{N}\}$ and the product has the form $M_i \times M_j \to M_{i+j}$. All monoids considered will be unital (with unit in M_0). Here, a graded monoid functor is a functor from commutative $\mathbb{F}_2$-algebras to graded monoids.

Definition 4.2.1.

(1) Let $\mathscr{M}$ denote the graded monoid functor defined on a commutative $\mathbb{F}_2$-algebra A by
$$\mathscr{M}_n(A) := \{\sum_{i=0}^{n} \alpha_i x^{2^i} | \alpha_0 = 1, \alpha_n \in A^\times\},$$

with product given by the composition of polynomials $(f,g) \mapsto f \circ g$ and unit $x \in \mathscr{M}_0(A)$.

(2) Let $\mathscr{M}^+$ denote the graded monoid functor defined by

$$\mathscr{M}^+_n(A) := \{\sum_{i=0}^{n} \alpha_i x^{2^i} | \alpha_0 = 1\},$$

with product given by composition, equipped with the monomorphism of monoid functors: $\mathscr{M} \hookrightarrow \mathscr{M}^+$ which forgets the invertibility condition.

Proposition 4.2.2.

(1) *The graded monoid functor $\mathscr{M}$ is represented by the connected graded bialgebra $\mathscr{B}$, where*

$$\mathscr{B}_n = \mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}, \alpha_n^{\pm 1}]$$

and $\mathscr{M}_n(\mathscr{B}_n)$ contains the universal element $\sum_{i=0}^{n} \alpha_i x^{2^i}$.

(2) *For each $n \in \mathbb{N}$, the commutative algebra $\mathscr{B}_n$ is graded, where $|\alpha_i| = 1 - 2^i$, so that the expression $\Sigma_i \alpha_i x^{2^i}$ is homogeneous of degree 1, when x is given degree 1.*

(3) *The inclusion of graded monoid functors $\mathscr{M} \hookrightarrow \mathscr{M}^+$ is represented by a sub graded bialgebra $\mathscr{B}^+ \hookrightarrow \mathscr{B}$, where $\mathscr{B}^+_n \cong \mathbb{F}_2[\alpha_1, \ldots, \alpha_n]$.*

(4) *For each $n \in \mathbb{N}$, the algebra $\mathscr{B}^+_n$ is a graded coconnected algebra (concentrated in negative degrees) and $\mathscr{B}_n \cong \mathscr{B}^+_n[\alpha_n^{-1}]$.*

Proof. Straightforward. (A key point is that the image of $\alpha_n \in \mathscr{B}_n$ under a diagonal $\Delta_{i,j}$ is an invertible element of $\mathscr{B}_i \otimes \mathscr{B}_j$.) $\square$

Remark 4.2.3.

(1) To fix the conventions used in defining the bialgebra structure, consider $\Delta_{1,1} : \mathscr{B}_2 \to \mathscr{B}_1 \otimes \mathscr{B}_1$, which is the morphism of $\mathbb{F}_2$-algebras $\mathbb{F}_2[\alpha_1, \alpha_2^{\pm 1}]$ which is defined by

$$\begin{array}{rcl} \alpha_1 & \mapsto & 1 \otimes \alpha_1 + \alpha_1 \otimes 1 \\ \alpha_2 & \mapsto & \alpha_1^2 \otimes \alpha_1. \end{array}$$

(2) The bialgebra $\mathscr{B}^+$ determines the bialgebra $\mathscr{B}$ by the localization which corresponds to inverting the element α_n of $\mathscr{B}_n$, for each n.

(3) For each n, the algebra $\mathscr{B}^+_n$ is of finite type; for $n \geq 2$, this is not true of $\mathscr{B}_n$.

4.3. **Group completion.** There is a familiar ungraded monoid functor, which can be interpreted as having underlying functor $\mathscr{M}^+_\infty$, by passing to formal power series:

Definition 4.3.1. Let $\mathscr{M}^+_\infty$ be the monoid functor defined by

$$\mathscr{M}^+_\infty(A) = \{\sum_{i=0}^{\infty} \alpha_i x^{2^i} | \alpha_0 = 1\},$$

with product given by the composition of formal power series.

Remark 4.3.2. This is a group functor, since composition inverses can be constructed working with formal power series.

The following is well-known:

Proposition 4.3.3. *The functor $\mathscr{M}^+_\infty$ is represented by the Hopf algebra $\mathscr{A}^* \cong \mathbb{F}_2[\xi_i | i \geq 0, \xi_0 = 1]$, the dual Steenrod algebra.*

Notation 4.3.4. Let Monoid denote the category of monoids (with unit) and $\mathrm{Monoid}^{\mathrm{gr}}$ the category of $\mathbb{N}$-graded monoids (with unit).

There is a forgetful functor $\mathcal{O} : \mathrm{Monoid}^{\mathrm{gr}} \to \mathrm{Monoid}$, which sends a graded monoid $(M_n | n \in \mathbb{N})$ to the monoid $\amalg_n M_n$.

Proposition 4.3.5. *The functor* $\mathcal{O} : \mathrm{Monoid}^{\mathrm{gr}} \to \mathrm{Monoid}$ *admits a right adjoint,* $\gamma : \mathrm{Monoid} \to \mathrm{Monoid}^{\mathrm{gr}}$, *which associates to a monoid N the graded monoid γN with $(\gamma N)_s = N$ for all s and structure morphisms induced by the product of N.*

Proof. Straightforward. □

This allows the following connection to be made between the dual Steenrod algebra and the graded bialgebra $\mathscr{B}^+$.

Proposition 4.3.6. *There is a morphism of monoid functors*

$$\mathcal{O}(\mathscr{M}^+) \to \mathscr{M}^+_\infty,$$

induced by considering a polynomial as a formal power series.

The adjoint morphism $\mathscr{M}^+ \to \gamma \mathscr{M}^+_\infty$ is induced by the morphism of graded bialgebras $\gamma \mathscr{A}^* \to \mathscr{B}^+$ (where γ is the analogue for bialgebras of the functor defined above), which has components

$$\theta_n : \mathbb{F}_2[\xi_i] \to \mathscr{B}^+_n \cong \mathbb{F}_2[\alpha_1, \ldots, \alpha_n]$$

$\xi_i \mapsto \alpha_i$, *for* $1 \leq i \leq n$, *and* $\xi_i \mapsto 0$, *for* $i > n$.

Proof. Clear. □

Remark 4.3.7. The morphism $\mathcal{O}(\mathscr{M}^+) \to \mathscr{M}^+_\infty$ can be viewed as a group completion of the monoid $\mathcal{O}(\mathscr{M}^+)$.

4.4. **Change of generators for $\mathscr{B}$.** There is a second standard choice of generators for the algebra $\mathscr{B}_n$ (for each $n \in \mathbb{N}$). In the following, by convention, $\alpha_0 = 1$.

Notation 4.4.1. Let $Q_{n,i}$ denote the element $\frac{\alpha_i}{\alpha_n}$ (for $0 \leq i \leq n$) of $\mathscr{B}_n$. (For other integers i, $Q_{n,i}$ is taken to be zero.) Hence $|Q_{n,i}| = 2^n - 2^i$.

Lemma 4.4.2. *For $n \in \mathbb{N}$, there is an isomorphism of algebras:*

$$\mathscr{B}_n \cong \mathbb{F}_2[Q_{n,0}^{\pm 1}, Q_{n,1}, \ldots, Q_{n,n-1}].$$

The diagonal $\Delta_{1,1} : \mathscr{B}_2 \to \mathscr{B}_1 \otimes \mathscr{B}_1$ is the morphism of $\mathbb{F}_2$-algebras determined by

$$\begin{array}{rcl} Q_{2,0} & \mapsto & Q_{1,0}^2 \otimes Q_{1,0} \\ Q_{2,1} & \mapsto & Q_{1,0} \otimes Q_{1,0} + Q_{1,0}^2 \otimes 1. \end{array}$$

Proof. Straightforward. □

Remark 4.4.3. In [Sin83], Singer uses invariant theory to construct a graded coalgebra Γ, where $\Gamma_n \cong \mathbb{F}_2[Q_{n,0}^{\pm 1}, \ldots, Q_{n,n-1}]$. The coproducts of the generators correspond to the coproducts of the Dyer-Lashof algebra (Cf. [Sin83, (2.14)]).

Proposition 4.4.4. *The graded coalgebra Γ has the structure of a graded bialgebra.*

Proof. A generalization of Lemma 4.4.2. □

Remark 4.4.5. The graded bialgebra Γ contains the sub-bialgebra Γ^-, which is defined by $\Gamma^-_n := \mathbb{F}_2[Q_{n,0}, \ldots, Q_{n,n-1}]$; this defines a sub-bialgebra $\mathscr{B}^-$ of $\mathscr{B}$.

4.5. **The coalgebras $\mathscr{B}$, $\mathscr{B}^+$ are quadratic.** In this section it is shown that the underlying coalgebras of $\mathscr{B}$ and $\mathscr{B}^+$ are quadratic. (This result can also be deduced from the results of [Sin05, Section 2].)

Proposition 4.5.1. *The underlying graded coalgebras of $\mathscr{B}^+$ and of $\mathscr{B}$ are cogenerated in degree one.*

Proof. A straightforward localization argument shows that it is sufficient to show that $\mathscr{B}^+$ is cogenerated in degree one. Moreover, it suffices to prove that, for each $n \geq 2$, the diagonal $\Delta_{n-1,1} : \mathscr{B}^+_n \to \mathscr{B}^+_{n-1} \otimes \mathscr{B}^+_1$ is a monomorphism.

The diagonal $\Delta_{n-1,1} : \mathbb{F}_2[\alpha_1, \ldots, \alpha_n] \to \mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}] \otimes \mathbb{F}_2[\alpha_1]$ is determined by

$$\begin{array}{rcl} \alpha_1 & \mapsto & 1 \otimes \alpha_1 + \alpha_1 \otimes 1 \\ \alpha_s & \mapsto & \alpha_{s-1}^2 \otimes \alpha_1 + \alpha_s \otimes 1 \\ \alpha_n & \mapsto & \alpha_{n-1}^2 \otimes \alpha_1, \end{array}$$

where $1 \leq s < n$.

We require to prove that the images α'_i of the elements α_i are algebraically independent. By composing with the projection $p : \mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}] \otimes \mathbb{F}_2[\alpha_1] \twoheadrightarrow \mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}]$ induced by $\mathbb{F}_2[\alpha_1] \to \mathbb{F}_2$, $\alpha_1 \mapsto 0$, it is straightforward to show that the elements $\alpha'_1, \ldots, \alpha'_{n-1}$ are algebraically independent.

Suppose that there exists a nonzero polynomial $f \in \mathbb{F}_2[\alpha'_1, \ldots, \alpha'_{n-1}][x]$ such that $f(\alpha'_n) = 0$, considered as an element of $\mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}] \otimes \mathbb{F}_2[\alpha_1]$ and choose such an f of minimal degree (in x). Since $p(\alpha'_n) = 0$, one has $p(f(\alpha'_n)) = p(f_0) = 0$. It follows that $f_0 = 0$, hence $f(x)$ can be replaced by $g(x) := f(x)/x$, which is again non-trivial and which satisfies $g(\alpha'_n) = 0$. This contradicts the hypothesis that the degree of f is minimal. $\square$

Notation 4.5.2. Let n be a positive integer and $1 \leq a \leq n$ be an integer. Write

(1) $j_n : \mathscr{B}^+_{n-1} \hookrightarrow \mathscr{B}^+_n$ for the canonical inclusion of algebras $\mathbb{F}_2[\alpha_1, \ldots, \alpha_{n-1}] \hookrightarrow \mathbb{F}_2[\alpha_1, \ldots, \alpha_n]$;
(2) $p_a : (\mathscr{B}^+_1)^{\otimes n} \twoheadrightarrow (\mathscr{B}^+_1)^{\otimes n-1}$ for the projection of algebras induced by the augmentation of the a^{th} factor;
(3) $\delta_a : \mathscr{B}^+_n \to (\mathscr{B}^+_1)^{\otimes n-1}$ for the composite of the iterated coproduct with p_a.

Lemma 4.5.3. *Let n be a positive integer and $1 \leq a \leq n$ be an integer.*

(1) *The composite $\mathscr{B}^+_{n-1} \xrightarrow{j_n} \mathscr{B}^+_n \xrightarrow{\delta_a} (\mathscr{B}^+_1)^{\otimes n-1}$ is the iterated coproduct.*
(2) *The kernel of $\delta_a : \mathscr{B}^+_n \to (\mathscr{B}^+_1)^{\otimes n-1}$ is the ideal $\alpha_n \mathscr{B}^+_n$.*

Proof. The first statement follows by considering the natural transformation which is represented by the composite morphism. In particular, this identification shows that the composite is a monomorphism. The second statement follows, by considering the decomposition as $\mathscr{B}^+_{n-1}$-modules $\mathscr{B}^+_n \cong \mathscr{B}^+_{n-1} \oplus \alpha_n \mathscr{B}^+_n$ given by using the inclusion j_n. $\square$

To avoid notational confusion, write $\mathbb{F}_2[u_1, \ldots, u_n]$ for the algebra $(\mathscr{B}^+_1)^{\otimes n}$, where u_i corresponds to the generator α_1 of the ith tensor factor; write α'_i for the image in $(\mathscr{B}^+_1)^{\otimes n}$ of the generator α_i.

Theorem 4.5.4. *The underlying coalgebras of $\mathscr{B}$ and of $\mathscr{B}^+$ are quadratic.*

Proof. By a localization argument (clearing fractions), it is sufficent to show that $\mathscr{B}^+$ is quadratic. Moreover, by induction upon n, it is sufficient to show that, for $n \geq 3$, the intersection of $\mathscr{B}^+_{n-1} \otimes \mathscr{B}^+_1$ and $\mathscr{B}^+_1 \otimes \mathscr{B}^+_{n-1}$ in $(\mathscr{B}^+_1)^{\otimes n}$ is equal to $\mathscr{B}^+_n$.

The diagonals $\Delta_{n-1,1}$ and $\Delta_{1,n-1}$ can be calculated explicity; for notational clarity, different sets of generators $\{\beta_i\}$ and $\{\gamma_i\}$ for $\mathscr{B}^+_{n-1}$ are used. By definition,

the morphism $\Delta_{n-1,1} : \mathscr{B}_n^+ \to \mathscr{B}_{n-1}^+ \otimes \mathscr{B}_1^+ \cong \mathbb{F}_2[\beta_1, \ldots, \beta_{n-1}] \otimes \mathbb{F}_2[u_n]$ is determined by the coefficients of the composition

$$(x + u_n x^2) \circ (x + \beta_1 x^2 + \beta_2 x^4 + \ldots + \beta_{n-1} x^{2^{n-1}}),$$

so $\Delta_{n-1,1}$ is given by:

$$\begin{array}{rcl} \alpha_1 & \mapsto & 1 \otimes u_n + \beta_1 \otimes 1 \\ \alpha_s & \mapsto & \beta_{s-1}^2 \otimes u_n + \beta_s \otimes 1 \quad (1 < s < n) \\ \alpha_n & \mapsto & \beta_{n-1}^2 \otimes u_n. \end{array}$$

Similarly, $\Delta_{1,n-1} : \mathscr{B}_n^+ \to \mathscr{B}_1^+ \otimes \mathscr{B}_{n-1}^+ \cong \mathbb{F}_2[u_1] \otimes \mathbb{F}_2[\gamma_1, \ldots, \gamma_{n-1}]$ is determined by the coefficients of the composition

$$(x + \gamma_1 x^2 + \gamma_2 x^4 + \ldots + \gamma_{n-1} x^{2^{n-1}}) \circ (x + u_1 x^2),$$

so that $\Delta_{1,n-1}$ is given by:

$$\begin{array}{rcl} \alpha_1 & \mapsto & u_1 \otimes 1 + 1 \otimes \gamma_1 \\ \alpha_s & \mapsto & u_1^{2^{s-1}} \otimes \gamma_{s-1} + 1 \otimes \gamma_s \quad (1 < s < n) \\ \alpha_n & \mapsto & u_1^{2^{n-1}} \otimes \gamma_{n-1}. \end{array}$$

Hence there are isomorphisms of subalgebras of $\mathbb{F}_2[u_1, \ldots, u_n]$:

$$\begin{array}{rcl} \mathscr{B}_{n-1}^+ \otimes \mathscr{B}_1^+ & \cong & \mathbb{F}_2[\alpha'_1, \ldots, \alpha'_{n-1}, u_n] \\ \mathscr{B}_1^+ \otimes \mathscr{B}_{n-1}^+ & \cong & \mathbb{F}_2[\alpha'_1, \ldots, \alpha'_{n-1}, u_1]. \end{array}$$

Let X belong to the intersection; considering X in $\mathbb{F}_2[\alpha'_1, \ldots, \alpha'_{n-1}, u_1]$, by subtracting an appropriate element of $\mathbb{F}_2[\alpha'_1, \ldots, \alpha'_{n-1}]$, we may suppose that X is divisible by u_1. If $X = 0$, there is nothing to prove; otherwise, by an inductive argument based on the degree, it is sufficient to show that X is divisible by α'_n, using the fact that, if $X = Y\alpha'_n$, then Y belongs to the intersection (this follows from the identifications $\alpha'_n = \beta_{n-1}^2 u_n = u_1^{2^{n-1}} \gamma_{n-1}$).

Considering X as an element of $\mathscr{B}_{n-1}^+ \otimes \mathscr{B}_1^+$, X belongs to the kernel of $\delta_1 \otimes \mathscr{B}_1^+$, hence by Lemma 4.5.3, X is divisible by $\beta_{n-1} = u_1^{2^{n-2}} \ldots u_{n-1}$, in particular, is divisible by u_{n-1}.

Using u_{n-1}-divisibility and considering X as an element of $\mathscr{B}_1^+ \otimes \mathscr{B}_{n-1}^+$, Lemma 4.5.3 (applied with respect to $\mathscr{B}_1^+ \otimes \partial_{n-2}$) implies that X is divisible by $\gamma_{n-1} = u_2^{2^{n-2}} \ldots u_{n-1}^2 u_n$, so that the element $X' := X/\beta_{n-1}$ of $\mathscr{B}_{n-1}^+ \otimes \mathscr{B}_1^+$ is divisible by $u_{n-1}u_n$.

Repeating this argument for $X' \in \mathscr{B}_{n-1}^+ \otimes \mathscr{B}_1^+$, which is u_{n-1}-divisible, it follows that X' is divisible by β_{n-1} and by u_n. Hence, X is divisible by $\alpha'_n = \beta_{n-1}^2 u_n$, as required. $\square$

5. Admissibility and duality for $\mathscr{B}$

The main result, Theorem 5.3.1, of this section shows that the underlying graded coalgebra of $\mathscr{B}$ is strictly self-dual.

5.1. **Admissibility for $\mathscr{B}$.** The graded bialgebra $\mathscr{B}$ has an internal degree; the underlying graded vector space of $\mathscr{B}_1$ is isomorphic to $\mathbb{F}_2[\mathbb{Z}]$ and the subspace $\mathscr{B}_1^+$ is isomorphic to $\mathbb{F}_2[\mathbb{N}]$ as graded vector spaces. (Here the *cohomological* grading has been adopted.)

Definition 5.1.1. Let

(1) $\mathcal{S} \subset \mathbb{Z} \times \mathbb{Z}$ denote the subset $\{(i, j) | i \geq 2j\}$;
(2) let $\mathcal{S}'$ denote the complement $\mathcal{S}' := \{(i, j) | i < 2j\}$.

Notation 5.1.2. Let $\partial : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ denote the bijection $(i,j)\mapsto(i+2,j+1)$, which restricts to bijections

$$\begin{array}{rcl} \partial : \mathcal{S} & \stackrel{\cong}{\to} & \mathcal{S} \\ \partial : \mathcal{S}' & \stackrel{\cong}{\to} & \mathcal{S}'. \end{array}$$

Lemma 5.1.3. *Let (i,j) be an element of $\mathcal{S}$ then $(i,j)=\partial^j(e,0)$, where $e = i-2j\geq 0$ is the excess, so that $(e,0)\in\mathcal{S}$.*

The following provides an ordering result which is useful for constructing admissible bases (compare [PP05, Chapter 4, Section 1]).

Lemma 5.1.4. *Let $(i,j)\in\mathcal{S}$ and $(l,m)\in\mathcal{S}'$ such that $i+j=l+m$, then $l<i$.*

Proposition 5.1.5. *The graded coalgebra $\mathscr{B}$ is $(\mathcal{I},\mathcal{S})$-admissible.*

Proof. Recall that $\mathscr{B}_2 \hookrightarrow \mathscr{B}_1\otimes\mathscr{B}_1 \cong \mathbb{F}_2[x^{\pm1},y^{\pm1}]$ is induced by localization of the algebra morphism $\mathbb{F}_2[\alpha_1,\alpha_2]\to\mathbb{F}_2[x,y]$ given by

$$\begin{array}{rcl} \alpha_1 & \mapsto & x+y \\ \alpha_2 & \mapsto & x^2y. \end{array}$$

To prove the result, it suffices to show that, for each $(i,j)\in\mathcal{S}$, there exists a unique element $h_{i,j}$ of $\mathbb{F}_2[\alpha_1,\alpha_2^{\pm1}]$ such that $h_{i,j}$ maps to x^iy^j modulo $\mathbb{F}_2[\mathcal{S}']$. Since $\alpha_2^jh_{(e,0)}$ satisfies this condition, where $e=i-2j$, it suffices to construct the elements $h_{(e,0)}$. The existence and the unicity of these elements can be shown by standard methods, based on Lemma 5.1.4 and the left lexicographical order on $\mathbb{Z}\times\mathbb{Z}$. (An explicit construction of the elements $h_{e,0}$ is given by the following result.) □

Proposition 5.1.6. *The elements $h_{(e,0)}$, for $e\in\mathbb{N}$, are defined recursively by*

$$\begin{array}{rcl} h_{(0,0)} & = & 1 \\ h_{(e+1,0)} & = & \alpha_1h_{(e,0)}+\alpha_2h_{(e-2,0)}, \end{array}$$

where $h_{(k,0)}$ is taken to be zero if $k<0$.

Moreover, the image of $h_{(e,0)}$ in $\mathbb{F}_2[x,y]$ is

$$x^e+\sum_u\binom{u-1}{2u-e-1}x^{e-u}y^u.$$

Proof. The proof is by induction; for $e\leq 2$, the result is immediate, hence we may suppose that $e>2$. Observe that the binomial coefficient is zero for $2u-e-1<0$; if $2u-e-1\geq 0$, the excess corresponding to the monomial $x^{e-u}y^u$ is $(e-u)-2u = e-3u$, which is negative (since e is non-negative and $e-2u\leq -1$). Hence the given polynomial has the required form. Thus, to establish the result, by unicity, it suffices to show that the given monomials $h_{(e,0)}$ have the stated images.

The image of $\alpha_1h_{(e,0)}$ in $\mathbb{F}_2[x,y]$ is

$$x^{e+1}+\Big\{\sum_u\binom{u-1}{2u-e-1}x^{e-u+1}y^u\Big\}+x^ey+\sum_v\binom{v-2}{2v-e-3}x^{e-v+1}y^v,$$

where the second sum has been reindexed by $v=u+1$.

Similarly, the image of $\alpha_2h_{(e-2,0)}$ is

$$x^ey+\sum_v\binom{v-2}{2v-e-1}x^{e-v+1}y^v.$$

Using the relations

$$\binom{u-2}{2u-e-2}+\binom{u-2}{2u-e-3}=\binom{u-1}{2u-e-2}=\binom{u-1}{2u-(e+1)-1}$$

completes the inductive step. □

Corollary 5.1.7. *The underlying quadratic coalgebra of* $\mathscr{B}$ *is isomorphic to* $\langle \mathbb{Z}; \mathcal{S}, f\rangle$, *where the function* $f : \mathcal{S} \times \mathcal{S}' \to \mathbb{F}_2$ *is given on pairs* $((i,j),(l,m)) \in \mathcal{S} \times \mathcal{S}'$ *such that* $i + j = l + m$ *by*

$$f((i,j),(l,m)) = \binom{m-j-1}{2(m-j)-e-1} = \binom{m-j-1}{m+j-l-1},$$

where $e = i - 2j$*; on pairs such that* $i + j \neq l + m$*, the function is trivial.*

In particular, the quadratic coalgebra $\langle \mathbb{Z}; \mathcal{S}, f\rangle$ *is dualizable.*

Proof. The proof of Proposition 5.1.5 indicates that there is a relation

$$f((i,j),(l,m)) = f((i-2j,0),(l-2j,m-j))$$

and $i - 2j$ is the excess e. Moreover, Proposition 5.1.6 shows that

$$f((e,0),(u,v)) = \binom{u-1}{2u-e-1}$$

if $u + v = e$ and is trivial otherwise.

The first identity follows immediately (using the fact that $i + j = l + m$ is equivalent to the corresponding condition $u + v = e$, where $u = l - 2j$, $v = m - j$).

To show that $\langle \mathbb{Z}; \mathcal{S}, f\rangle$ is dualizable, it is sufficient to show that, for a fixed $(l,m) \in \mathcal{S}'$, the set of $(i,j) \in \mathcal{S}$ such that $f((i,j),(l,m)) \neq 0$ is finite.

It suffices to consider $(i,j) \in \mathcal{S}$ such that $i + j = l + m$. The binomial coefficient

$$\binom{m-j-1}{m+j-l-1}$$

is zero if $m - j - 1 < 0$ or if $m + j - l - 1 < 0$. It follows that there is a finite interval of values of j for which the binomial coefficient can be non-trivial. This implies the result.

□

Corollary 5.1.8. *The quadratic coalgebra* $\mathscr{B}^+$ *is* $(\mathbb{N}, \mathcal{S}_{\mathbb{N}})$*-admissible.*

Proof. An immediate consequence of Proposition 2.6.2. (The result can be seen directly.) □

5.2. **The weak coPBW property.**

Proposition 5.2.1. *The* $(\mathbb{Z}, \mathcal{S})$*-admissible quadratic bialgebra* $\mathscr{B}$ *satisfies the weak coPBW property.*

Proof. (Indications.) It is straightforward to see that it is sufficient to prove the result for $\mathscr{B}^+$, which is $(\mathbb{N}, \mathcal{S}_{\mathbb{N}})$-admissible. Here standard inductive calculations apply, generalizing the methods used in the proof that $\mathscr{B}^+$ is quadratic and that it is $(\mathbb{N}, \mathcal{S}_{\mathbb{N}})$-admissible. (Cf. [Sin83, Lemma 2.7], which establishes an analogous result.) □

Remark 5.2.2.
(1) A detailed proof is not given, since this result is related to the fact that the universal Steenrod algebra has a PBW basis. (See the next section.)
(2) The existence of such a basis is intimately related to the argument used by Singer in [Sin83, Proposition 8.1] to prove that the dual of the opposite of the Lambda algebra is a quotient coalgebra of Γ (compare Proposition 2.7.4).

5.3. **Self-duality.** Recall that the coalgebra underlying the bialgebra $\mathscr{B}$ is isomorphic to the quadratic coalgebra $\langle \mathbb{Z}; \mathcal{S}, f\rangle$ and is dualizable; the transpose dual coalgebra is $\langle \mathbb{Z}; \mathcal{S}', f^!\rangle$.

Theorem 5.3.1. *The quadratic coalgebra $\langle \mathbb{Z}; \mathcal{S}, f\rangle$ is strictly self-dual with respect to the bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$, $n \mapsto 1-n$.*

Proof. It is elementary to check that σ restricts to a bijection between $\mathcal{S}$ and $\mathcal{S}'$ (note that σ^2 is the identity). Hence, it suffices to show that, for $((i,j),(l,m)) \in \mathcal{S}\times\mathcal{S}'$, there is an equality

$$f((i,j),(l,m)) = f((\sigma(l,m),\sigma(i,j)) = f((1-l,1-m),(1-i,1-j)).$$

Clearly it is sufficient to restrict to the case $i+j = l+m$. The right hand side is given by the binomial coefficient

$$\binom{(1-j)-(1-m)-1}{(1-j)+(1-m)-(1-i)-1} = \binom{m-j-1}{i-m-j}.$$

This is equal to $f((i,j),(l,m))$, since $i-m-j = l-2j$ (using $i+j = l+m$) and $m+j-l-1 = (m-j-1)-(l-2j)$. □

6. Applications to the universal Steenrod algebra

6.1. **The universal Steenrod algebra.** The universal Steenrod algebra was introduced by May in [May70] (under the name big Steenrod algebra) in his general algebraic approach to the construction of Steenrod operations; here we consider only the mod-2 case. The terminology universal Steenrod algebra was introduced by Lomonaco, since the Dyer-Lashof algebra, the opposite of the Lambda algebra and the Steenrod algebra all appear as sub-quotients of the algebra. No algebraic universal property is claimed.

The following presentation suffices here:

Definition 6.1.1. (Cf. [BCL05], for example.) The universal Steenrod algebra over $\mathbb{F}_2$ is the quadratic algebra $\mathscr{Q}$ generated by elements $\{y^i | i \in \mathbb{N}\}$, with y_i of internal degree i, subject to the generalized Adem relations

$$y_{2k-1-n}y_k = \sum_j \binom{n-1-j}{j} y_{2k-1-j}y_{k+j-n}.$$

Lemma 6.1.2. *The generalized Adem relations are equivalent to:*

$$y_u y_v = \sum_m \binom{v-m-1}{v+m-u-1} y_{u+v-m}y_m,$$

where $u < 2v$.

Definition 6.1.3. Let $\widetilde{\mathscr{A}}$ denote the quotient of $\mathscr{Q}$ by the ideal generated by $\{y_j | j < 0\}$. Equivalently, $\widetilde{\mathscr{A}}$ is the quadratic algebra generated by the elements $\{Sq^i | i \in \mathbb{N}\}$ (with internal degree $|Sq^i| = i$) subject to the Adem relations (without the condition $Sq^0 = 1$). (This algebra is denoted $\mathcal{B}$ in [Sin05].)

6.2. **Self-duality and the relationship between $\mathscr{B}$ and $\mathscr{Q}$.** Transpose duality for admissible quadratic coalgebras has an obvious counterpart for quadratic algebras, as does self-duality:

Definition 6.2.1. Let $\mathcal{I}$ be a set and $\mathcal{S} \subset \mathcal{I}\times\mathcal{I}$ be a subset. A quadratic algebra $\{\mathbb{K}[\mathcal{I}]; R\}$ is $(\mathcal{I},\mathcal{S})$-admissible if and only if the quadratic coalgebra $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ is $(\mathcal{I},\mathcal{S})$-admissible; in this case, the quadratic algebra $\{\mathbb{K}[\mathcal{I}]; R\}$

(1) is dualizable if and only if $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ is dualizable;
(2) is strictly self-dual if and only if $\langle \mathbb{K}[\mathcal{I}]; R\rangle$ is strictly self-dual.

Recall that the bialgebra $\mathscr{B}$ is isomorphic (as a graded coalgebra) to the admissible coalgebra $\langle \mathbb{Z}; \mathcal{S}, f\rangle$, in the notation of Section 5.

Theorem 6.2.2.

(1) *The universal Steenrod algebra $\mathscr{Q}$ is isomorphic to the quadratic algebra associated to the quadratic coalgebra $\langle \mathbb{Z}; \mathcal{S}', f^{!}\rangle$*
(2) *The universal Steenrod algebra $\mathscr{Q}$ is dualizable and is strictly self-dual.*

Proof. For the first statement, compare the coefficients of the generalized Adem relations (in the form given in Lemma 6.1.2) with the function f.

The second statement follows from Theorem 5.3.1. □

Remark 6.2.3. (1) It is known that the universal Steenrod algebra can be constructed as the quadratic algebra $\{\mathbb{F}_2[\mathbb{Z}]; \Gamma_2\}$, where Γ_2 is the degree two part of Singer's graded bialgebra Γ [Lom90]. This presentation of the universal Steenrod algebra [Lom90] can be recovered using self-duality.

Quadratic self-duality can be deduced from [Lom92] (where a weaker reciprocity result is established, since the considerations are motivated by the study of Koszul, finite-type objects). Moreover, the calculation of the diagonal cohomology of the universal Steenrod algebra [Lom97] is essentially equivalent to this theorem, as is the main result of [Lom06].

(2) The reciprocity results of Section 3.3, Proposition 3.3.1 and Proposition 3.3.2, apply to the bialgebra $\mathscr{B}$ to recover the results of Lomonaco [Lom92].

Recall that each algebra $\mathscr{B}_n^+$ is of finite type (with respect to the internal degree), hence taking the vector space dual (that is the dual in the naïve sense) yields a quadratic algebra.

Corollary 6.2.4. [Sin05, Theorem 1.2, Proposition 2.11] *The quadratic algebra $\widetilde{\mathscr{A}}$ is the dual of the quadratic bialgebra $\mathscr{B}^+$.*

Remark 6.2.5. Analogous results hold at odd primes; this will be developed elsewhere.

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